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ON SEVERAL POINTS IN THE THEORY OF THE GROUPS OF A FINITE ORDER.*

By DR. G. A. MILLER.

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1. *Definition.* The most prominent condition which the operators of a group must satisfy is that the product of any two of them is equivalent to a single operator contained in the group. When the operators of the group are represented by substitutions it is not necessary to add any additional condition, *i. e.* a set of substitutions which includes every substitution that is obtained by multiplying any two of the set together or by squaring any one of them forms a group, provided that no two of the substitutions are identical.

It is however not customary to call *any* set of operators that satisfies the given condition a group : *e. g.* the three numbers 0, 1, −1 clearly satisfy this condition while they would not generally be said to form a group when they are combined by multiplication. Some writers seem to make the definition of group so general as to include this case,† but the theory of the groups of operators which satisfy this general definition without also satisfying the more restricted ones remains to be developed.

In addition to the given conditions the operators of a group are usually required to satisfy the associative law, and the laws that the product of any two of them is completely determined by the operators and the method of combining them and that from the equations $ab=ac$, $\alpha\beta=\gamma\beta$ it must follow respectively that $b=c$, $\alpha=\gamma$; where a, b, c, α, β, γ represent any operators of the group.

While the method of combining the operators of a group is generally called multiplication yet it should not be inferred that any restrictions are imposed upon this method except those given above, so that the term multiplication in this connection merely implies some definite law of combination : *e. g.* if we combine the following n numbers, 0, 1, 2, 3, ..., $n-1$, by adding any one to each of them, their positive remainders with respect to modulus n will determine the cyclical group of order n and every cyclical group can be represented in this way. If we combine these numbers by multiplication, taking their positive remainders according to the same modulus, they do *not* form a group in the usual sense of this term. It is thus clear that the group property is not inherent in a set of operators but, that it exists, in part, in the law of combination.

As may be inferred from the heading, these remarks relate to the groups of a finite order. This restriction seemed appropriate for this occasion since my investigations have been confined to this class of groups. In recent years the groups of an infinite order have received a great deal of attention. Among the investigators of the discontinuous groups of an infinite order Poincaré, Klein and Fricke are the most prominent while Sophus Lie is preëminent among the many investigators of continuous groups. After this glance at these two sturdy

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*Read before the *Oliver Mathematical Club* of Cornell University as initiatory paper, February, 1898.

†Cf. Klein's Ikosaeder, page 5; Miss Scott's Modern Analytical Geometry, page 281; etc.

offsprings of our subject we proceed to give a short sketch of its development.

Historical Sketch. While the group concept appears early in the development of mathematics yet it was not until substitution groups began to be studied that this concept began to receive considerable attention and that its fundamental importance began to be recognized. Writers do not agree in regard to the founder of the theory of substitutions. Some ascribe this honor to Lagrange, others to Abel, others to Galois, and still others to Cauchy. In recent years a considerable number, under the leadership of Burkhardt,* have been contending that the origin of the theory of substitutions is found in an Italian work bearing the title "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto*, di Paolo Ruffini," Bologna, 1799.

This work contains some fundamental concepts such as transitivity, intransitivity, primitivity, etc., that are generally attributed to the later works of Cauchy. Its proof that the general equation whose degree exceeds four is not algebraically solvable lacks rigor. The first vigorous proof of this important theorem, which engaged the attention of mathematicians for centuries, was given by Abel and it was published in the first volume of Crelle, the oldest German mathematical paper still living. As this proof was partly based upon the theory of substitutions it drew considerable attention to this subject. This seems to be the main reason why some writers, Hagen for example, call Abel one of the founders of the theory of substitutions.†

The next great impulse to the study of this theory was given by that very remarkable young mathematician Galois, who proved that every algebraic equation belongs to a group and that its solvability by the extraction of roots depends upon the factors of composition of this group. This discovery united the theory of equations very closely with the theory of substitution groups. The French mathematician Jordan has been especially prominent in pointing out results which depend upon this relation. The subject is one of great difficulty and it will probably furnish a fertile field of investigation for many years to come.

The claims that Cauchy was the founder of the theory of substitutions seem to be based upon the fact that he was the first to write extensively on this subject apart from any direct applications. His *Exercices d' Analyse*, vol. 3, (1844), contains the first systematic treatment of this theory and contributed very largely towards making it known to a larger class of students. Although some of the concepts which have generally been attributed to him have recently been traced to earlier writings yet there remains much which is undoubtedly due to Cauchy and some of these facts are of fundamental importance.

In recent years there has been developing an offspring from substitution groups, which resembles the parent more closely than the two mentioned above. I refer to the operation or abstract groups. In the first volume of the *American Journal of Mathematics* Cayley calls attention to the fact that a group is an ab-

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**Schlomilch's Zeitschrift*, 1892, supplement, page 159.

†*Synopsis der Mathematik*, page 281.

stract concept and that it should be defined "by means of the laws of combination of its symbols." A few years later Dyck published several articles in the *Mathematische Annalen* in which he laid the foundation of a part of the theory of abstract groups.

As many of the properties of these groups can be readily studied by means of substitution groups their development has given a new impulse to the study of substitution groups. This is the modern impulse and it is to be hoped that it will lead to great activity in this line. The bulk of the recent publications on groups of a finite order have been along the lines of abstract groups and these groups promise to remain a fertile field of investigation for a number of years at least.

Standing Problems. While knowledge is cosmopolitan and scientific discoveries belong to the world rather than to a particular nation yet it does not behoove a great nation or even a great university to be satisfied to borrow all its scientific facts from the rest of the world. Such a state of affairs indicates stagnation and is evidence either of inability or of want of patriotism. It also deprives the students of the inspiration and joy which attend the discovery of important scientific facts that tend to contribute something towards enriching all future generations.

It is evident that the standing problems of a comparatively new subject can be reached more readily and are generally less difficult than those of older subjects. Hence the newer subjects are the more inviting to the young investigator. While our modern mathematical journals do a great deal towards aiding the student to find desirable unsolved problems yet some of the simpler ones are solved at such a rapid rate that it is scarcely possible to keep in touch with a large portion of the region in which these discoveries are being made.

It has been observed for a long time that the problem to find all the substitution groups of any degree would be of the greatest importance to algebra.* This problem is far from a complete solution. It has been solved for all the degrees less than eleven but the methods employed are not suitable for very large degrees and they throw very little light on the general problem. The general theorems on this subject are still few, and the progress that has been made in recent years under the inspiration of the great prize of the Paris Institute gives evidence of the great difficulty of the problem rather than any hope of its early solution.

In recent years all the simple groups of the finite continuous groups have been found. It would be a great step forward if all the simple groups of the groups of a finite order could also be found. Several new infinite systems of such groups have recently been discovered by Professors Moore and Burnside, and by Dr. Dickson but the progress towards the complete solution of this problem is exceedingly slow. Any discoveries that throw light on this subject are of great interest.

As problems of somewhat smaller interest in themselves we may mention

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*Cf. Serret's *algebra superieure*, 3rd edition, page 256.

the proof of the existence or non-existence of simple groups of an odd order or of order $p^{\alpha} q^{\beta}$, p and q being prime numbers ; the superior limit of transitivity of primitive groups that do not contain the alternating group ; the simplification of the methods of proving the solvability or the insolvability of a group, etc.

REPLY TO PROFESSOR FISK'S CRITICISM OF A CERTAIN FEATURE OF NICHOLSON'S CALCULUS.

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In the March number of the *Bulletin* is a brief review of my Calculus, by Professor Fiske, of which the following is an extract :

"In another note (A_3) at the end of the work the author critizes the grounds assigned by Byerly and by Rice and Johnston for making $d(dx)=0$. He contends that the differential of dx is zero, because dx as a variable is independent of x. This, of course, is not sound. If a variable y is independent of another variable x, it is true that we may still write

$$dy=\frac{dy}{dx}dx;$$

but the coefficient of dx is not a partial derivative, and dy, therefore, instead of being zero is indeterminate. In order that $d(dx)$ may be zero, we must assume that dx takes the same value for all values of x. This assumption, however, does not prevent our varying dx from one instant to another in a perfectly arbitrary manner."

As the question involved is an interesting and important one, and believing that Professor Fiske had not fairly presented my discussion of the point at issue, I wrote a brief reply to the above criticism, and sent it to the *Bulletin* for publication. Several weeks thereafter my reply was returned to me without publication and with the following additional stricture :

"The author fails to realize that if $dy=0$ when x goes from x to $x+dx$ then y is not completely independent of x, but has such a dependence that it does not alter when x alters."

The question involved is not whether dx is a quantity whose total differential is 0, but whether it is a quantity whose differential *with respect to x* is 0.

Of course if x and y are two variables which are independent of each other, and dx be the total differential of the one and dy that of the other, and $\frac{dy}{dx}$ is understood to mean the ratio of these differentials, $\frac{dy}{dx}$ is not 0 but indeterminate, as Professor Fiske says. Or again, under the same hypothesis, "if $dy=0$